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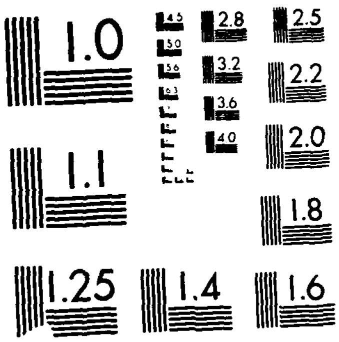

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AN APPROACH FOR CONSTRUCTING FAMILIES OF HOMOGENIZED EQUATIONS FOR PERIODIC MEDIA:

I: An Integral Representation and Its Consequences

R. C. Morgan

and

I. Babuška
Institute for Physical Science and Technology
and
Department of Mathematics
University of Maryland
College Park, Maryland 20742

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The paper, which is the first in a series of two, presents an approach which allows us to derive a family of homogenization approaches and assess the accuracy of any homogenization in the relation of given input data.

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AN APPROACH FOR CONSTRUCTING FAMILIES OF HOMOGENIZED EQUATIONS FOR PERIODIC MEDIA:

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R.C. Morgan

and

I. Babuška[1]
Institute for Physical Science and Technology and
Mathematics Department
University of Maryland
College Park, MD 20742

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Abstract.

The paper, which is the first in a series of two, presents an approach which allows us to derive a family of homogenization approaches and assess the accuracy of any homogenization in the relation of given input data.

1. Introduction.

The study of periodic media is one application of partial differential equations that have highly oscillatory, periodic coefficients. Essentialy, the problem is to solve the elliptic differential equation

$$(1)\qquad -\sum_{p,q=1}^{n} \frac{\partial}{\partial x_p}\left[a_{pq}\left(\frac{x}{h}\right)\frac{\partial u^h}{\partial x_q}(x)\right] + a_0\left(\frac{x}{h}\right)u^h(x) = f(x)$$

on $\Omega \subset \mathbb{R}^n$ with prescribed boundary conditions, in which a_{pq} and a_0 are real-valued 2π-periodic functions and h is a positive number that is small in comparison with the diameter of the domain Ω.

The problem is to get the solution of (1) for relatively (to what?) small h. There is large available mathematical literature which addresses the behavior of the solution of (1) as $h \to 0$. We mention here for example [3], [8], [9] and survey [20].

One of the main applications of differential equations of the type (1) is in the field of composite materials. Here the aim is to replace the composite by homogeneous materials with the bulk material properties. For various aspects we refer to [1], [2], [11], [12], [18]. A brief history is given in [2]. The accuracy of such replacement depends, of course, on the goals of the analysis. Hence many approaches are used in applications. The most obvious approach, namely to use asymptotic analysis for $h \to 0$, is not always applicable because h is given and cannot be changed and because of particular aims of the analysis. In a similar vein, when numerically solving the problem (1), one faces essential difficulties of how to represent the microstructure of the composite materials. This difficulty falls into the class of solution of elliptic equations with rough coefficients. For various aspects of this problem, we refer to [6], [7].

As was said above, various approaches can be and are used for solving (1), approaches which often give very different results, see e.g. [10]. In addition some of these approaches, although in principle well described, are leading to large technical difficulties because a lot of symbol manipulation is needed. The use of symbolic manipulation on computers does not simplify these difficulties too much.

This paper presents and thoroughly analyzes an approach which is directed to overcome the various major difficulties mentioned above:

a) It allows the design of an entire class of "homogenization" formulations and judge the accuracy and reliability of any homogenization approach. It also allows the specification of the class of problems (e.g. loads) for which a homogenization approach is applicable. In addition, it leads to a hierarchal construction of the homogenization formulations.

b) The implementation is completely numerical, and allows adaptive modeling (selection of the equations).

We will address here only the problem with $\Omega = \mathbb{R}^n$ although very important features of the solution occur near the boundary when Ω is a bounded domain. These problems have a special character and will not be addressed here. Some comments will be made in section 5.

We will assume that

i) $\Omega = \mathbb{R}^n$

ii) $a_0(x) \geq \gamma_0 > 0$

iii) $f \in L_2(\mathbb{R}^n)$

and that the problem is elliptic and self-adjoint.

The restriction of our analysis to a single differential equation is of a technical character only, as the ideas are also applicable to a system of equations, which would arise in elasticity problems, for example. The main idea of the approach, under the assumptions stated above, is based on the result that the solution u^h of (1) can be written in the form

$$u^h(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(t)\phi(\frac{x}{h}, h, t)e^{it\cdot x}dt$$

in which $\hat{f}$ is the Fourier transform of f and $\phi(y,h,t)$ is a function, which is 2π-periodic in y and analytic in h and t, and which solves the differential equation

$$(3) \qquad -e^{-iht\cdot y} \sum_{p,q=1}^{n} \frac{\partial}{\partial y_p}\left[a_{pq}(y)\frac{\partial}{\partial y_p}(\phi(y)e^{iht\cdot y})\right] + h^2 a_0(y)\phi(y) = h^2$$

on $\{y \in \mathbb{R}^n : |y_p| < \pi\}$. Other representations of u^h that are related to (2), but developed in a different context, can be found in [8] and [16].

By taking various expansions of Φ with respect to h and t, we can express approximately the solution u^h in terms of solutions of auxiliary partial differential equations with constant coefficients, or even pseudodifferential equation, or alternatively, we can design a system of "ansatz" functions to be used in a finite element model. Considering the error estimates associated with the expansions of ϕ allows us to design an adaptive method of selecting a "model" that would yield an approximate solution, whose accuracy meeds a prescribed tolerance. These ideas are more fully discussed in [6], where we introduced a method to systematically derive numerical, computer oriented methods for an approximation of u^h.

In this paper, we concentrate our attention on the representation (2), whereas we will make a thorough analysis of $\phi(y,h,t)$ in [15]. Consequently, the properties of ϕ that are used in this paper will be stated without proof. The integral in (2) is defined as a Bochner integral of $H^1_{-\nu}(\mathbb{R}^n)$-valued function ($H^1_{-\nu}(\mathbb{R}^n)$ is defined in the next section). As a simple application of (2), we will given an alternate proof, in section 5, of the classical homogenization result (the limit of u^h as $h \to 0$).

This paper and [15] are based on the first author's Ph.D. thesis [14], in which additional details and references can be found.

2. Notation and Statement of the Problem.

For $j = 0, 1$, and for any $\nu \in \mathbb{R}$, define the weighted Sobolev space $H^j_\nu(\mathbb{R}^n)$ to be the completion of $C^\infty_0(\mathbb{R}^n)$ (the complex-valued C^∞-functions that have compact support on $\mathbb{R}^n$), with respect to $\|\cdot\|_{j,\nu}$, where

$$\|u\|^2_{j,\nu} = \int_{\mathbb{R}^n} \sum_{|\alpha|\le j} |D^\alpha u(x)|^2 e^{2\nu|x|}dx.$$

(For $x \in \mathbb{R}^n$, $|x| \equiv |x_1| + \ldots + |x_n|$.) We will use $H^j(\mathbb{R}^n)$ and $\|\cdot\|_j$ to denote the standard Sobolev space and norm on $\mathbb{R}^n$ (i.e., when $\nu = 0$). Next, we introduce the Sobolev spaces of periodic functions for which

$$S \equiv \{y \equiv (y_1, \ldots, y_n) \in \mathbb{R}^n : |y_k| < \pi \text{ for } k = 1, \ldots, n\}$$

is the fundamental period. For $j = 0, 1$, we denote the standard Sobolev norm on S by $\|\cdot\|_{j,S}$, and we define $H^j_{per}(S)$ to be the completion, with respect to the norm $\|\cdot\|_{j,S}$, of the complex-valued C^∞-functions on $\mathbb{R}^n$ that are 2π-periodic in each coordinate variable.

Let a_{pq} $(p, q = 1, \ldots, n)$ and a_0 be real-valued, 2π-periodic, L_∞-functions defined on $\mathbb{R}^n$. Furthermore, assume $a_{qp} = a_{pq}$ and assume that there exist positive constants γ_0 and γ_1 such that

(4) $$\begin{cases} a_0(x) \geq \gamma_0 \quad \text{and} \\ \sum\limits_{p,q=1}^{n} a_{pq}(x)\zeta_q\bar{\zeta}_p \geq \gamma_1 \sum\limits_{p=1}^{n} |\zeta_p|^2 \quad \text{for all} \quad \zeta_p \in \mathbb{C}, \end{cases}$$

almost everywhere on $\mathbb{R}^n$. For each $h > 0$, define

$$\Psi(h)[u,v] = \int_{\mathbb{R}^n} \left\{ \sum_{p,q=1}^{n} a_{pq}(\tfrac{x}{h}) \frac{\partial u}{\partial x_q}(x) \overline{\frac{\partial v}{\partial x_p}(x)} + a_0(\tfrac{x}{h}) u(x) \overline{v(x)} \right\} dx.$$

An immediate consequence of the conditions imposed on the coefficients a_0 and a_{pq} is that there exists a constant C, independent of $h > 0$, such that

(5) $$\begin{cases} |\Psi(h)[u,v]| \leq C\|u\|_1 \|v\|_1 \quad \text{and} \\ |\Psi(h)[v,v]| \geq \min\{\gamma_0, \gamma_1\} \|v\|_1^2 \end{cases}$$

for all u and v in $H^1(\mathbb{R}^n)$. Then, according to the Lax-Milgram theorem, for each $h > 0$ and each $f \in L_2$, there exists a unique function $u^h \in H^1(\mathbb{R}^n)$ that satisfies

(6) $$\Psi(h)[u^h, v] = \int_{\mathbb{R}^n} f(x)\overline{v(x)} dx \quad \text{for all} \quad v \in H^1(\mathbb{R}^n),$$

because

(7) $$v \longmapsto \int_{\mathbb{R}^n} f(x)\overline{v(x)} dx$$

is a bounded linear functional on $H^1(\mathbb{R}^n)$.

Next, for each $h \in \mathbb{C}$ and $t \in \mathbb{C}^n$, define the sequilinear form $\Phi(h,t) : H^1_{per}(S) \times H^1_{per}(S) \to \mathbb{C}$ by

$$\Phi(h,t)[\phi,v] \equiv \int_S \left\{ \sum_{p,q=1}^{n} a_{pq}(y) \frac{\partial}{\partial y_p}(\phi(y)e^{iht\cdot y}) \frac{\partial}{\partial y_p}(\overline{v(y)}e^{-iht\cdot y}) + h^2 a_0(y)\phi(y)v(y) \right\} dy$$

Lemma 1. A neighborhood $\hat{G} \subset \mathbb{C}^{n+1}$, of $\mathbb{R}^{n+1}$ can be found such that for each $(h,t) \in \hat{G}$, there exists a unique function $\phi(\cdot,h,t) \in H^1_{per}(S)$ that satisfies

$$\Phi(h,t)[\phi(\cdot,h,t),v] = h^2\int_S \overline{v(y)}dy \quad \text{for all} \quad v \in H^1_{per}(S).$$

Furthermore, the mapping $(h,t) \in \hat{G} \longmapsto \phi(\cdot,h,t) \in H^1_{per}(S)$ is holomorphic, by which we mean that about each point in $\hat{G}$, the function $(h,t) \longmapsto \phi(\cdot,h,t)$ can be expanded in a power series, convergent in $H^1_{per}(S)$ and in which each coefficient is an element in $H^1_{per}(S)$.

For the most part, the proofs of statements concerning $\phi(\cdot,h,t)$ are omitted in this paper since we give a fairly complete analysis of $\phi(\cdot,h,t)$ in [15].

In section 4, we show that u^h admits the representation

$$(8) \qquad u^h(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \hat{f}(t)\phi(\tfrac{x}{h},h,t)e^{it\cdot x}dt,$$

in which $\hat{f}(t) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x)e^{-it\cdot x}dx$ and in which the integral is a Bochner integral of $H^1_{-\nu}(\mathbb{R}^n)$-valued functions. Our proof of (8) has as its first step the claim that for each $h > 0$ and $t \in \mathbb{R}^n$,

$$(9) \qquad x \longmapsto \phi(\tfrac{x}{h},h,t)e^{it\cdot x}$$

solves (6) when $f(x) = e^{it\cdot x}$. However, (9) is not an element of $H^1(\mathbb{R}^n)$, and for this choice of f, (7) is not a bounded linear functional on $H^1(\mathbb{R}^n)$. Consequently, we consider $\Psi(h)$ as a sesquilinear form on $H^1_{-\nu}(\mathbb{R}^n)\times H^1_{\nu}(\mathbb{R}^n)$ for (sufficiently small) positive numbers ν.

The main tool for analyzing $\Psi(h)$ is

Theorem 2. Let H_1 and H_2 be two complex Hilbert spaces with respective norms $\|\cdot\|_k$ and associated inner products $(\cdot,\cdot)_k$ for $k = 1,2$. Let $B[\cdot,\cdot]$ be a sesquilinear form defined on $H_1 \times H_1$ for which there exist positive constants M and γ such that

a) $|B[u,v]| \le M\|u\|_1\|v\|_2$ for all $u \in H_1$ and $v \in H_2$,

b) $\inf_{\substack{u\in H_1 \\ \|u\|_1=1}} \sup_{\substack{v\in H_2 \\ \|u\|_2\le 1}} |B[u,v]| \ge \gamma > 0$, and

c) $\sup_{u\in H_1} |B[u,v]| > 0$ for each $v \in H_2$, $v \ne 0$.

If $f \in H_2^*$, the space of bounded conjugate-linear functionals on H^2, then there exists a unique $u_0 \in H_1$ such that

d) $B[u_0,v] = f(v)$ for all $v \in H_2$, and

e) $\|u_0\|_1 \le \frac{1}{\gamma}\|f\|_{H_2^*}$.

A proof of theorem 2 in the case of real Hilbert spaces can be found in [4] (as Theorem 5.2.1). The method of proof in the complex case is essentially unchanged and thus will be omitted.

We now prove

Lemma 3. There exist positive constants ν_0, C, and γ such that for all $\nu \in (0,\nu_0)$ and all $h > 0$,

i) $|\Psi(h)[u,v]| \le C\|u\|_{1,-\nu}\|v\|_{1,\nu}$,

ii) $\inf_{\|u\|_{1,-\nu}=1} \sup_{\|v\|_{1,\nu}=1} |\Psi(h)[u,v]| \ge \gamma > 0$, and

iii) $\sup_{u\in H^1_{-\nu}(\mathbb{R}^n)} |\Psi(h)[u,v]| > 0$ for all $v \in H^1_\nu(\mathbb{R}^n)$ and $v \ne 0$.

The constants ν_0, C, and γ are independent of $h > 0$; however, γ depends on ν_0.

Proof. Statement (i) follows because $\Psi(h)$ has L_∞-coefficients. To prove statement (ii), define $(Tu)(x) \equiv u(x)e^{-2\nu|x|}$ for $u \in H^1_{-\nu}(\mathbb{R}^n)$. Then for

$\nu > 0$,

$$(10) \qquad \|Tu\|^2_{1,\nu} = \int_{\mathbb{R}^n}\left[\sum_{j=1}^{n}\left|\frac{\partial u}{\partial x_j}(x) - 2\nu\ \mathrm{sgn}(x_j)u(x)\right|^2 + |u(x)|^2\right]e^{-2\nu|x|}dx$$

$$\le 2(1+4n\nu^2)\|u\|^2_{1,-\nu}.$$

Now,

$$(11) \qquad \Psi(h)[u,Tu] = \int_{\mathbb{R}^n}\left\{\sum_{p,q=1}^{n} a_{pq}(\tfrac{x}{h})\frac{\partial u}{\partial x_q}(x)\overline{\frac{\partial(Tu)}{\partial x_p}(x)} + a_0(\tfrac{x}{h})u(x)\overline{v(x)}\right\}dx$$

$$= \int_{\mathbb{R}^n}\left\{\sum_{p,q=1}^{n} a_{pq}(\tfrac{x}{h})\frac{\partial u}{\partial x_q}(x)\overline{\frac{\partial u}{\partial x_p}(x)} + a_0(\tfrac{x}{h})u(x)\overline{u(x)}\right\}e^{-2\nu|x|}dx$$

$$= -2\nu\int_{\mathbb{R}^n}\left\{\sum_{p,q=1}^{n} a_{pq}(\tfrac{x}{h})\frac{\partial u}{\partial x_q}(x)\mathrm{sgn}(x_p)\overline{u(x)}\right\}e^{-2\nu|x|}dx$$

$$\equiv \Psi_1(h)[u] - 2\nu\Psi_2(h)[u].$$

A simple consequence of (4) is $\Psi_1(h)[u] \ge \min\{\gamma_0,\gamma_1\}\|u\|^2_{1,-\nu}$. We also have $|\Psi_2(h)[u]| \le c\|u\|^2_{1,-\nu}$ for some constant c, independent of h. Combining these two inequalities with (10) and (11) yields

$$|\Psi(h)[u,Tu]| \ge (\min\{\gamma_0,\gamma_1\} - 2c\nu)\|u\|^2_{1,-\nu} \ge \frac{\min\{\gamma_0,\gamma_1\} - 2c\nu}{\sqrt{2(1+4n\nu^2)}}\|u\|_{1,-\nu}\|Tu\|_{1,\nu}.$$

Consequently, there exist positive constants ν_0 and γ, which are independent of $h > 0$, for which $\Psi(h)[u,Tu] \ge \gamma\|u\|_{1,-\nu}\|Tu\|_{1,\nu}$ for all $\nu \in (0,\nu_0)$. This proves (ii).

Statement (iii) is proven in a similar manner. □

Throughout the remainder of this paper, we implicitly assume $\nu \in (0,\nu_0)$. Let $f \in L_2(\mathbb{R}^n)$; then

$$\left|\int_{\mathbb{R}^n} f(x)\overline{v(x)}dx\right| \le \|f\|_0\|v\|_{1,\nu} \quad \text{for all} \quad v \in H^1_\nu(\mathbb{R}^n).$$

Lemma 3, in conjunction with Theorem 2, now yields

<u>Theorem 4</u>. For each $h > 0$ and $f \in L_2(\mathbb{R}^n)$ there exists a unique function $u^h \in H^1_{-\nu}(\mathbb{R}^n)$ for which

(12) $$\Psi(h)[u^h,v] = \int_{\mathbb{R}^n} f(x)\overline{v(x)}dx \quad \text{for all} \quad v \in H^1_\nu(\mathbb{R}^n).$$

Furthermore, $\|u^h\|_{1,-\nu} \le \frac{1}{\gamma}\|f\|_0$.

There is no ambiguity in denoting the unique solutions of (6) and (12) by u^h because $H^1_\nu(\mathbb{R}^n) \subset H^1(\mathbb{R}^n) \subset H^1_{-\nu}(\mathbb{R}^n)$ implies they are the same function. However, (12) can be solved when f belongs to a broader class of functions than the class of L_2-functions; namely, when f belongs to the dual space of $H^1_\nu(\mathbb{R}^n)$. One such function is defined by $f(x) = e^{it\cdot x}$.

3. <u>Preliminaries</u>.

It is easy to prove

<u>Theorem 5</u>. There exists a constant C_0 independent of t, such that

$$\|\chi e^{it\times y}\|_{1,S} \le C_0(1+\|t\|)\|\chi\|_{1,S}$$

for al: $\chi \in H^1(S)$ and for all $t \in \mathbb{R}^n$, where $\|t\|^2 \equiv t_1^2 + \ldots + t_n^2$.

For each $h > 0$ and each $\omega \in \mathbb{Z}^n$, define

$$S(h,\omega) \equiv \{x \in \mathbb{R}^n : (\omega_j-1)\pi h < x_j < (\omega_j+1)\pi h, \quad j = 1,\ldots,n\}.$$

Note that $S(1,(0,\ldots,0)) = S$ is the fundamental domain for the periodic spaces.

Lemma 6. For each $h > 0$ and $\nu > 0$, there exists a constant $C_1(h,\nu)$ that remains bounded as $h \to 0$, such that for all $t \in \mathbb{R}^n$, and for any $\chi \in H^0_{per}(S)$,

i) $\|\chi(\frac{x}{h})e^{it\cdot x}\|_{0,-\nu} \le C_1(h,\nu)\|\chi\|_{0,S}$

and for any $\chi \in H^1_{per}(S)$,

ii) $\|\chi(\frac{x}{h})e^{it\cdot x}\|_{1,-\nu} \le (1+h^{-1})C_1(h,\nu)\|\chi e^{iht\cdot y}\|_{1,S}.$

Proof.

$$\|\chi(\tfrac{x}{h})e^{it\cdot x}\|^2_{1,-\nu} \le \sum_{\omega\in\mathbb{Z}^n}\int_{S(h,2\omega)}\left[\sum_{p=1}^{n}|\frac{\partial}{\partial x_p}(\chi(\tfrac{x}{h})e^{it\cdot x})|^2 + |\chi(\tfrac{x}{h})|^2\right]e^{-2\nu|x|}dx,$$

where $2\omega \equiv (2\omega_1,\ldots,2\omega_n)$. Making the substitution $\frac{x}{h} - y + 2\pi\omega$ in the integral over $S(h,2\omega)$ and using the periodicity of χ yields a constant $\tilde{c}$, independent of h, such that

$$\|\chi(\tfrac{x}{h})e^{it\cdot x}\|^2_{1,-\nu} = \sum_{\omega\in\mathbb{Z}^n}\left\{\int_S\left[h^{-1}\sum_{p=1}^{n}|\frac{\partial}{\partial y_p}(\chi(y)e^{iht\cdot y})e^{i2\pi h\omega\cdot t}|^2 + |\chi(y)|^2\right)\right.$$

$$\left.\cdot e^{-2\nu h|y+2\pi\omega|}h^n dy\right\}$$

$$\le \tilde{c}(1+h^{-2})\left[\sum_{\omega\in\mathbb{Z}^n}e^{-4\pi\nu h|\omega|}\right]h^n\|\chi e^{iht\cdot y}\|^2_{1,S}.$$

It is not difficult to prove that there is a constant c that is independent of h, such that

$$h^n\sum_{\omega\in\mathbb{Z}^n}e^{-4\pi\nu h|\omega|} \le ch^n(1+\int_{\mathbb{R}^n}e^{-r\pi\nu h|x|}dx).$$

Upon setting $C_1(h,\nu) \equiv \sqrt{\tilde{c}c(h^n+(2\pi\nu)^{-n})}$, (ii) follows.

A similar argument in which the contribution from the first order derivatives is ignored, produces (i). □

<u>Lemma 7</u>. For all $h > 0$ and $t \in \mathbb{R}^n$, we have

$$\|\phi(\cdot,h,t)\|_{0,S} \le \frac{(2\pi)^{n/2}}{\gamma_0}$$

and

$$\|\phi(\cdot,h,t)e^{iht\cdot y}\|_{1,S} \le C_2(h)$$

for some positive number $C_2(h)$.

<u>Proof</u>. It follows from (4) and Lemma 1 that

$$\gamma_1\int_S \sum_{p=1}^n \left|\frac{\partial}{\partial y_p}(\phi(y,h,t)e^{iht\cdot y})\right|^2 dy + \gamma_0 h^2\|\phi(\cdot,h,t)\|_{0,S}^2$$
$$\le \Phi(h,t)[\phi(\cdot,h,t),\phi(\cdot,h,t)]$$
$$\le (2\pi)^{n/2}h^2\|\phi(\cdot,h,t)\|_{0,S}.$$

The lemma now follows with $C_2(h) \equiv \dfrac{(2\pi)^{n/2}}{\min\{\gamma_0,\gamma_1 h^{-2}\}}$. □

4. <u>The representation of</u> u^h.

We begin with

<u>Theorem 8</u>. For each $h > 0$ and $t \in \mathbb{R}^n$,

i) $x \longmapsto \phi(\frac{x}{h},h,t)e^{it\cdot x}$ is in $H^1_{-\nu}(\mathbb{R}^n)$, and

ii) $\Psi(h)[\phi(\frac{x}{h},h,t)e^{it\cdot x},v] = \int_{\mathbb{R}^n} e^{it\cdot x}\overline{v(x)}dx$ for all $v \in H^1_\nu(\mathbb{R}^n)$.

Furthermore,

iii) $\|\phi(\frac{x}{h}, h, t)e^{it\cdot x}\|_{1,-\nu} \le \frac{1}{\gamma\nu^{n/2}}$.

Proof. Statement (i) follows from Lemmas 6 and 7, which imply $\|\phi(\frac{x}{h}, h, t)e^{it\cdot x}\|_{1,-\nu} \le (1+h^{-1})C_1(h,\nu)C_2(h)$. Assuming that (ii) is true, (iii) is a consequence of Theorem 2, Lemma 3, and the fact $|\int_{\mathbb{R}^n} e^{it\cdot x}\overline{v(x)}dx| \le \frac{1}{\nu^{n/2}}\|v\|_{1,\nu}$. The proof of (ii) is based upon determining the relationship between $\Psi(h)$ and $\Phi(h,t)$, and then using Lemma 1.

For each $h > 0$, there exists a locally finite, C^∞-partition of unity $\{\sigma_\omega(\cdot,h) : \omega \in \mathbb{Z}^n\}$ subordinated to $\{S(h,\omega) : \omega \in \mathbb{Z}^n\}$ such that $\sum_{\omega\in\mathbb{Z}^n}\sigma_\omega(\cdot,h)v$ converges to v in $H^1_\nu(\mathbb{R}^n)$ whenever $v \in H^1_\nu(\mathbb{R}^n)$. The basic requirement of the partition of unity is that $|\sigma_\omega(x,h)|$ and $|\frac{\partial\sigma_\omega}{\partial x_p}(x,h)|$ for $p = 1,\dots,n$ are uniformly bounded for $x \in \mathbb{R}^n$ and $\omega \in \mathbb{Z}^n$. Then $v_\omega(\cdot,h) \equiv \sigma_\omega(\cdot,h)v$ ahs compact support in $S(h,\omega)$, and for any $\chi \in H^1_{per}(S)$,

$$\Psi(h)[\chi(\tfrac{x}{h})e^{it\cdot x}, v] = \sum_{\omega\in\mathbb{Z}^n}\Psi(h)[\chi(\tfrac{x}{h})e^{it\cdot x}, v_\omega(x,h)]$$

$$\le \sum_{\omega\in\mathbb{Z}^n}\int_{S(h,\omega)}\left[\sum_{p,q=1}^{n} a_{pq}(\tfrac{x}{h})\frac{\partial}{\partial x_q}(\chi(\tfrac{x}{h})e^{it\cdot x})\overline{\frac{\partial v_\omega}{\partial x_p}(x,h)}\right.$$

$$\left. + a_0(\tfrac{x}{h})\chi(\tfrac{x}{h})e^{it\cdot x}\overline{v_\omega(x,h)}\right]dx$$

because $\Psi(h)$ is a continuous sesquilinear form on $H^1_{-\nu}(\mathbb{R}^n)\times H^1_\nu(\mathbb{R}^n)$, according to Lemma 3 (i). In an effort to deform the region of integration $S(h,\omega)$ into $S = S(1,(0,\dots,0))$ in each integral, make the substitution $\frac{x}{h} = y + \pi\tilde{\omega}$ for $x \in S(h,\omega)$, in which $\tilde{\omega}$ is the n-tuple of even integers, that is, derived from ω according to

$$\tilde{\omega}_p = \begin{cases} \omega_p & \omega_p \text{ even,} \\ \omega_p - 1, & \omega_p \text{ odd.} \end{cases}$$

Using the periodicity of a_0, $\{a_{pq} : p,q = 1,\ldots,n\}$, and χ, each integral in (13) becomes

$$(14) \qquad \int_{S(1,\omega-\tilde{\omega})} \left\{ h^{-2} \sum_{p,q=1}^{n} a_{pq}(y) \frac{\partial}{\partial y_q}(\chi(y)e^{iht\cdot(y+\pi\tilde{\omega})}) \overline{\frac{\partial v_\omega}{\partial x_p}(h(y+\pi\tilde{\omega}),h)} \right.$$

$$\left. + a_0(y)\chi(y)e^{iht\cdot(y+\pi\tilde{\omega})}\overline{v_\omega(h(y+\pi\tilde{\omega}),h)} \right\} h^n dy.$$

Next, for each $\omega \in \mathbb{Z}^n$, define

$$(15) \qquad v_\omega^0(y,h,t) \equiv v_\omega(h(y+\pi\tilde{\omega}),h)e^{-iht\cdot(y+\pi\tilde{\omega})}$$

for $y \in S(1,\omega-\tilde{\omega})$ and extend $v_\omega^0(\cdot,h,t)$ to all of $\mathbb{R}^n$ by 2π-periodicity. Since the support of $y \longmapsto v_\omega(h(y+\pi\tilde{\omega}),h)$ is contained in $S(1,\omega-\tilde{\omega})$, it follows that $v_\omega^0(\cdot,h,t) \in H^1_{per}(S)$. Using (15) to substitute for $v_\omega(h(y+\pi\tilde{\omega}),h)$ in (14) yields

$$\int_{S(1,\omega-\tilde{\omega})} \left\{ h^{-2} \sum_{p,q=1}^{n} a_{pq}(y) \frac{\partial}{\partial y_q}(\chi(y)e^{iht\cdot y}) \overline{\frac{\partial}{\partial y_q}(v_\omega^0(y,h,t)e^{-iht\cdot y})} \right.$$

$$\left. + \sigma a_0(y)\chi(y)\overline{v_\omega^0(y,h,t)} \right\} h^n dy.$$

Now, the domain of integration $S(1,\omega-\tilde{\omega})$ can be replaced with $S(1,(0,\ldots,0)) = S$, and consequently

$$(16) \qquad \Psi(h)[\chi(\tfrac{x}{h})e^{it\cdot x},v] = \sum_{\omega\in\mathbb{Z}^n} h^{n-2}\Phi(h,t)[\chi,v_\omega^0(\cdot,h,t)]$$

for all $v \in H^1_\nu(\mathbb{R}^n)$.

Noting Lemma 1, it is now a simple matter to prove (ii):

$$\Psi(h)[\phi(\tfrac{x}{h},h,t)e^{it\cdot x},v] = \sum_{\omega\in\mathbb{Z}^n} h^{n-2}\Phi(h,t)[\phi(\cdot,h,t),v^0_\omega(\cdot,h,t)]$$

$$= \sum_{\omega\in\mathbb{Z}^n} h^n \int_S \overline{v^0_\omega(y,h,t)}dy$$

$$= \int_{\mathbb{R}^n} e^{it\cdot x}v(x)dx$$

for all $v \in H^1_\nu(\mathbb{R}^n)$, since $v^0_\omega(\cdot,h,t) \in H^1_{per}(S)$. □

<u>Lemma 9</u>. For each $h > 0$, $t \in \mathbb{R}^n \longmapsto \phi(\frac{x}{h},h,t)e^{it\cdot x} \in H^1_{-\nu}(\mathbb{R}^n)$ is a continuous mapping.

<u>Proof</u>. The continuity of $t \longmapsto e^{itx} \in H^1_{-\nu}(\mathbb{R}^n)$ follows in a straightforward manner.

Upon setting $t = 0$ and $\chi = \phi(\cdot,h,t) - \phi(\cdot,h,\tau)$ in Lemma 6

$$\lim_{t\to\tau}\|\phi(\tfrac{x}{h},h,t) - \phi(\tfrac{x}{h},h,\tau)\|_{1,-\nu}$$

$$\le (1+h^{-1})C_1(h,\nu)\lim_{t\to\tau}\|\phi(\cdot,h,t) - \phi(\cdot,h,\tau)\|_{1,S}$$

$$= 0$$

follows from Lemma 1. □

For each $f \in L_2(\mathbb{R}^n)$, the Fourier transform of f is defined by

$$\hat{f}(t) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x)e^{-it\cdot x}dx.$$

The following notation will be used with the function f only (as it appears in (12)). For any $f \in L_2(\mathbb{R}^n)$ and $N > 0$, define $f_N \in L_2(\mathbb{R}^n)$ as the inverse Fourier transform of $\hat{f}|_{\{t:\|t\|\le N\}}$, i.e.

$$f_N(x) = \frac{1}{(2\pi)^{n/2}} \int_{\|t\|\le N} \hat{f}(t)e^{it\cdot x}dt \quad \text{and} \quad \hat{f}_N(t) = \begin{cases} \hat{f}(t), & \|t\| \le N \\ 0, & \|t\| > N. \end{cases}$$

Parseval's inequality implies

$$\lim_{N\to\infty}\|f - f_N\|_0 = \lim_{N\to\infty}\|\hat{f} - \hat{f}_N\|_0 = 0.$$

Next, for each $N > 0$, define $u^h(\cdot;N) \in H^1_{-\nu}(\mathbb{R}^n)$ by

$$u^h(x;N) = \frac{1}{(2\pi)^{n/2}} \int_{\|t\|\le N} \hat{f}(t)\phi(\frac{x}{h},h,t)e^{it\cdot x}dt$$

$$\left[= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}_N(t)\phi(\frac{x}{h},h,t)e^{it\cdot x}dt\right],$$

in which the integral is to be interpreted as a Bochner integral of $H^1_{-\nu}(\mathbb{R}^n)$-valued function (cf. [17]). In order to show that the integral has such a meaning, we need to show that the integrand is strongly measurable and that $t \longmapsto \|\hat{f}(t)\phi(\frac{x}{h},h,t)e^{it\cdot x}\|_{1,-\nu}$ is Lebesgue integrable over $\{t : \|t\| \le N\}$. This integrability condition is satisfies as a result of (iii) of Theorem 8 and because $\hat{f}$ is an L_1-function on $\{t : \|t\| \le N\}$. By strong measurability of the integrand, we mean there exists a sequence of simple functions $t \longmapsto w_k(\cdot,t) \in H^1_{-\nu}(\mathbb{R}^n)$ such that

$$\lim_{k\to\infty}\|\hat{f}(t)\phi(\frac{x}{h},h,t)e^{it\cdot x} - w_k(x,t)\|_{1,-\nu} = 0. \tag{17}$$

It is easy to construct such a sequence (and we do so in the proof of Theorem 10 below) using Lemma 9 and the measurability of f. Furthermore, if the sequence of simple functions satisfies

$$\lim_{k\to\infty} \int_{\|t\|\le N} \|\hat{f}(t)\phi(\frac{x}{h},h,t)e^{it\cdot x} - w_k(x,t)\|_{1,-\nu} = 0 \tag{18}$$

then

$$\int_{\|t\|\le n} \hat{f}(t)\phi(\tfrac{x}{h},h,t)e^{it\cdot x}dt = \lim_{k\to\infty} \int_{\|t\|\le N} w_k(\cdot,t)dt \quad \text{in} \quad H^1_{-\nu}(\mathbb{R}^n),$$

in which the integral of a simple function is defined in the standard manner.

Theorem 10. For each $N > 0$,

$$\psi(h)[u^h(\cdot;N),v] = \int_{\mathbb{R}^n} f_N(x)\overline{v(x)}dx \quad \text{for all} \quad v \in H^1_\nu(\mathbb{R}^n).$$

Proof. The essence of this proof is that a sequence of simple functions satisfying (17) and (18) can be chosen so that each value of each simple function is of the form $c\phi(\frac{x}{h},h,\tau)e^{i\tau\cdot x}$ for some $\tau \in \{t : \|t\| \le N\}$ and for some complex number c. With this being the case, Theorem 8 (ii) can be used to evaluate $\Psi(h)[u,v]$ whenever u is the integral of one of these simple functions.

To begin, note that $H^1_\nu(\mathbb{R}^n) \subset L_1(\mathbb{R}^n)$. Consequently, $\hat{v}$ is continuous (cf. [15]) when $v \in H^1_\nu(\mathbb{R}^n)$, and it follows from Theorem 8 that

$$\Psi(h)[\phi(\tfrac{x}{h},h,\tau)e^{i\tau\cdot x},v] = (2\pi)^{n/2}\overline{\hat{v}(\tau)}$$

is well defined for each $\tau \in \mathbb{R}^n$.

Set $\Omega \equiv \{t \in \mathbb{R}^n : \|t\| \le N\}$. Now select a sequence of simple functions $s_k : \Omega \longrightarrow \mathbb{C}$ and a sequence of collections $\{\Omega_{k,j} : j = 0,\dots,M_k\}$ of measurable subsets of Ω for $k = 1,2,\dots$ and for which the following is true. For each k:

a) $\Omega = \bigcup_{j=0}^{M_k} \Omega_{k,j}$, $\Omega_{k,j} \cap \Omega_{k,\ell} = \phi$ for $j \ne \ell$;

b) for $j = 1,\dots,M_k$, if t and τ are in $\Omega_{k,j}$, then $\|\phi(\frac{x}{h},h,t)e^{it\cdot x} - \phi(\frac{x}{h},h,t)e^{it\cdot x}\|_{1,-\nu} < 2^{-k}$, and $|\hat{v}(t) - \hat{v}(\tau)| < 2^{-k}$; and

c) s_k is constant on $\Omega_{k,j}$ for $j = 0,\dots,M_k$, with $s_k(t) = 0$ for $t \in \Omega_{k,0}$, $s_k \longrightarrow \hat{f}_N$ pointwise a.e. on Ω, and $|s_k(t)| \le |\hat{f}_N(t)|$ a.e. on Ω.

Next, for each k, pick a $\tau^{k,j} \in \Omega_{k,j}$ for each $j = 1,\dots,M_k$, and define

$$w_k(x,t) = \begin{cases} 0, & t \in \Omega_{k,0}, \\ \phi(\frac{x}{h}, h, \tau^{k,j})e^{i\tau^{k,j}\cdot x}, & t \in \Omega_{k,j}, \end{cases}$$

and

$$r_k(t) = \begin{cases} 0, & t \in \Omega_{k,0} \\ \hat{v}(\tau^{k,j}), & t \in \Omega_{k,j}. \end{cases}$$

A consequence of the preceding construction and of Theorem 8 (iii) is

$$\begin{aligned} &\|\hat{f}_N(t)\phi(\tfrac{x}{h}, h, t)e^{it\cdot x} - s_k(t)w_k(x,t)\|_{1,-\nu} \\ &\le |\hat{f}_N(t) - s_k(t)|\frac{1}{\gamma\nu^{n/2}} + |\hat{f}_N(t)|2^{-k} \\ &\le \left[\frac{1}{\gamma\nu^{n/2}} + 2^{-k}\right]|\hat{f}_N(t)|. \end{aligned}$$

It follows from the first inequality that

$$\lim_{k\to\infty}\|\hat{f}_N(t)\phi(\tfrac{x}{h}, h, t)e^{it\cdot x} - s_k(t)w_k(\cdot,t)\|_{1,-\nu} = 0 \quad \text{a.e.}$$

Furthermore, the Lebesgue dominated convergence theorem and the second inequality imply

$$\lim_{k\to\infty}\int_\Omega \|\hat{f}_N(t)\phi(\tfrac{x}{h}, h, t)e^{it\cdot x} - s_k(t)w_k(\cdot,t)\|_{1,-\nu}dt = 0.$$

Consequently, $u^h(\cdot,N) = \lim_{k\to\infty}(2\pi)^{-n/2}\int_{\|t\|\le N} s_k(t)w_k(\cdot,t)dt$ in $H^1_{-\nu}(\mathbb{R}^n)$.

Finally, the continuity of $u \longmapsto \Psi(h)[u,v]$ (Lemma 3); the definitions of

$s_k, w_k,$ and r_k; and Parseval's equality implies

$$\Psi(h)[u^h(\cdot;N),v] = \lim_{k\to\infty}(2\pi)^{-n/2}\left[\int_{\|t\|\le N} s_k(t)w_k(\cdot,t)dt, v\right]$$

$$= \lim_{k\to\infty}\int_{\|t\|\le N} s_k(t)r_k(t)dt$$

$$= \int_{\|t\|\le N} \hat{f}_N(t)\overline{\hat{v}(t)}dt$$

$$= \int_{\mathbb{R}^n} f_N(x)\overline{v(x)}dx$$

for all $v \in H^1_\nu(\mathbb{R}^n)$, because $\hat{f}_N(t) = 0$ for $\|t\| > N$. □

It is now a simple matter to prove the main result of this paper.

<u>Theorem 11</u>. Suppose $h > 0$ and $f \in L_2(\mathbb{R}^n)$. Let $u^h \in H^1_{-\nu}(\mathbb{R}^n)$ be the solution of (12). Then

(19) $$u^h(x) = \lim_{N\to\infty}(2\pi)^{-n/2}\int_{\|t\|\le N} \hat{f}(t)\phi(\tfrac{x}{h},h,t)e^{it\cdot x}dt \quad \text{in} \quad H^1_{-\nu}(\mathbb{R}^n),$$

where, for each N, the integral is defined as a Bochner integral of $H^1_{-\nu}(\mathbb{R}^n)$-valued functions.

<u>Proof</u>. A consequence of Theorem 10 is that

(20) $$\Psi(h)[u^h - u^h(\cdot;N),v] = \int_{\mathbb{R}^n}(f(x)-f_N(x))\overline{v(x)}dx$$

for all $v \in H^1_\nu(\mathbb{R})$, from which the inequality

(21) $$\|u^h - u^h(\cdot;N)\|_{1,-\nu} \le \frac{1}{\gamma}\|f - f_N\|_0,$$

is easily derived (cv. Theorem 2 and Lemma 3). Then (19) follows. □

Actually, (19) converges in $H^1(\mathbb{R}^n)$ even though each integral is defined only as a function in $H^1_{-\nu}(\mathbb{R}^n)$. The reasoning that allows us to identify the unique solutions of (6) and (12) also yields $u^h(\cdot;N) \in H^1(\mathbb{R}^n)$ and the fact that (20) is valid for all $v \in H^1(\mathbb{R}^n)$. Now, the Lax-Milgram theorem and (5) imply

$$\|u^h - u^h(\cdot;N)\|_1 \le \frac{1}{\min\{\gamma_0,\gamma_1\}}\|f - f_N\|_0.$$

5. Homogenization.

In this section we derive first the classical result of homogenization which states that u^h converges as h tends to 0, to a function that is the solution of a constant-coefficient partial differential equation. This is an example of analyzing u^h through (19) and an analysis of $\phi(\cdot,h,t)$.

According to Lemma 1, it is possible to expand $\phi(\cdot,h,t)$ in powers of h, for each $t \in \mathbb{R}^n$. Consequently, we can write

(22) $$\phi(\cdot,h,t) = \phi_0(\cdot,t) + \phi_1(\cdot,t)h + \dots .$$

The functions $\{\phi_j(\cdot,t) : j = 0,1,\dots\}$ can be determined by expanding (3) in powers of h and substituting (22). Here, we are interested in only the constant term, and solving for it yields

$$\phi_0(\cdot,t) = g_0(t) = \frac{1}{\sum_{p,q=t}^{n} A_{pq}t_q t_p + A_0},$$

where A_0 and $\{A_{pq} : p,q = 1,\dots,n\}$ are derived from the periodic coefficients a_0 and $\{a_{pq} : p,q = 1,\dots,n\}$ and certain auxiliary functions. Complete details are given in [13], where proofs of the properties of $\phi(\cdot,h,t)$ that are sted in the following lemma can also be found.

Lemma 12. There exist positive constants θ and G_0, and continuous functions $g_0 : \mathbb{R}^n \to (0,\infty)$ and $G_1 : \{(h,t) \in \mathbb{R}^{n+1} : 0 \le \theta h(1+\|t\|) < 1\} \to (0,\infty)$ such that

i) $g_0(t) \le \dfrac{G_0}{1+\|t\|^2}$, and

ii) $\|\phi(\cdot,h,t) - g_0(t)\|_{1,S} \le G_1(h,t)h$ for each $h \ge 0$ and $t \in \mathbb{R}^n$ that satisfy $\theta h(1+\|t\|) < 1$.

A consequence of (i) is that we can define functions u_0 and $u_0(\cdot;N)$, for $N > 0$, in $H^2(\mathbb{R}^n)$ by

$$u_0(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \hat{f}(t)g_0(t)e^{it\cdot x}dt$$

$$u_0(x;N) = (2\pi)^{-n/2}\int_{\|t\|<N} \hat{f}(t)g_0(t)e^{it\cdot x}dt \tag{23}$$

whenever $f \in L_2(\mathbb{R}^n)$ (cf. [11]). Note that $u_0(\cdot;N) = u_0$ with $f = f_N$. We have $\hat{u}_0 = \hat{f}g_0$ and $(u_0(\cdot;N))\hat{} = \hat{f}_N g_0$, and then Parseval's equality implies

$$\|u_0 - u_0(x;N)\|_0 \le G_0\|f - f_N\|_0. \tag{24}$$

The integrals that define u_0 and $u_0(\cdot;N)$ are to be interpreted as Lebesgue integrals of numerical-valued functions. However, we want to interpret $u_0(\cdot;N)$ as an integral of $H^1_{-\nu}(\mathbb{R}^n)$-valued functions. In the appendix, we show that the integral in (23) can be interpreted in both ways, in an unambiguous and consistent manner. Furthermore, note that u_0 is the solution of the constant coefficient differential equation

$$(25) \qquad -\sum_{p,q=1}^{n} A_{pq} \frac{\partial^2 u_0}{\partial x_p \partial x_q}(x) + A_0 u_0(x) = f(x).$$

We can now prove the classical result in homogenization.

Theorem 13. Let $f \in L_2(\mathbb{R}^n)$. Let $u^h \in H^1_{-\nu}(\mathbb{R}^n$ be the solution of (12) for $h > 0$, and let $u_0 \in H^1(\mathbb{R}^n) \subset H^1_{-\nu}(\mathbb{R}^n)$ be the solution of (25). The $\lim_{h\to 0}\|u^h - u_0\|_{0,-\nu} = 0$.

Proof. For each $N > 0$, we have

$$(26) \qquad \|u^h - u_0\|_{0,-\nu} \le \|u^h - u^h(\cdot;N)\|_{0,-\nu} + \|u^h(\cdot;N) - u_0(\cdot;N)\|_{0,-\nu} + \|u_0(\cdot;N) - u_0\|_{0,-\nu}.$$

Let $\varepsilon > 0$ be given. It follows from (21) and (24) that we can choose an $N > 0$ that makes each of the first and third terms on the right-hand side of (26) smaller than ε, uniformly in h. Next, a consequence of Lemmas 6 and 12 is that there exists $h_0 \in (0, \frac{1}{\theta(1+N)})$ such that for all $h \in (0, h_0)$,

$$|u^h(\cdot;N) - u_0(\cdot;N)\|_{0,-\nu} \le (2\pi)^{-n/2} \int_{\|t\|\le N} |\hat{f}(t)|\,\|(\phi(\tfrac{x}{h}, h, t) - g_0(t))e^{it\cdot x}\|_{0,-\nu} dt$$

$$\le (2\pi)^{-n/2} C_1(h,\nu) h \int_{\|t\|\le N} |\hat{f}(t)| G_1(h,t) dt$$

$$< \varepsilon. \qquad \square$$

Above we have used expansion (22) of the function $\Phi(\cdot,h,t)$. Function $\Phi(\cdot,h,t)$ is the solution of the elliptic problem (3) with h and t being parameters. We can find $\Phi(\cdot,h,t)$ numerically e.g. by the finite element method for various h and t and then make various expansions or approximations by numerical approaches. The a posteriori error analysis will lead also to an assessment of the accuracy of the used homogenization.

We can use various approximations of $\Phi(\cdot,h,t)$ to derive the homogenization approach. Let us mention a few. For more, see [5].

a) For given a we can approximate $\Phi(\cdot,h,t)$ so that

$$\Phi(\cdot,h,t) \cong \sum \varphi_k(\cdot)\psi_k(t),$$

where $\psi_k(t)$ are rational functions in t. Every $\psi_k(t)$ then is the symbol of the homogenized equation

$$L_1(\cdot)u = L_2(\cdot)u.$$

b) We can also use the above approximation of $\Phi(\cdot,h,t)$ as an "ansatz" for the solution and derive equations for $\psi_k(t)$ via finite element method and energy minimization principle. This approach is essentially equivalent to the derivation of special elements for "rough" problems.

c) Using other approximations we can obtain pseudodifferential equations.

d) Any homogenization approach can and has to be understood as an approximation of $\Phi(\cdot,h,t)$. Hence it is possible to assess the accuracy of a homogenization approach by comparing it with $\Phi(\cdot,h,t)$.

By the above mentioned approach a system of second or higher order elliptic equations can be derived. This system will usually be of singularly perturbed type and the solution will show boundary larger when Ω is bounded. We can use various "ansatzes" of various types in different areas of the domain, especially in the boundary region By this way we can also obtain reliable solutions in the neighborhood of the boundary.

6. Appendix.

Define $w(x,t) = (w\pi)^{n/2}\hat{f}(t)g_0(t)e^{it\cdot x}$ where $f \in L_2(\mathbb{R}^n)$ and g_0 is defined in Lemma 13. According to (22), $u_0(\cdot;N) \in H^2(\mathbb{R}^n)$ is defined by

$$u_0(x;N) = \int_{\|t\|\le N} w(x,t)dt$$

as an integral of numerical-valued functions. Since

$$\begin{cases} t \longmapsto w(\cdot,t) \text{ is strongly measurable in } H^1_{-\nu}(\mathbb{R}^n), \text{ and} \\ \int_{\|t\|\le N} \|w(\cdot,t)\|_{1,-\nu}dt \le (2\pi)^{n/2}G_0\nu^{-n/2}\int_{\|t\|\le N} |\hat{f}(t)|dt < \infty, \end{cases}$$

it follows that

$$W = \int_{\|t\|\le N} w(\cdot,t)dt$$

can be defined as an integral of $H^1_{-\nu}(\mathbb{R}^n)$-valued functions. We want to prove

<u>Lemma 14</u>. $u_0(\cdot;N) = W$ as a function in $H^1_{-\nu}(\mathbb{R}^n)$.

<u>Proof</u>. It suffices to show that $x \longmapsto u_0(x;N)e^{-\nu|x|}$ and $x \longmapsto W(x)e^{-\nu|x|}$ generate the same generalized function. It follows from (27) that a sequence $t \longmapsto w_k(\cdot,t) \in H^1_{-\nu}(\mathbb{R}^n)$ of simple functions can be chosen so that

$$\lim_{k\to\infty}\|w_k(\cdot,t) - w(\cdot,t)\|_{1,-\nu} = 0 \quad \text{a.e.}$$

$$\lim_{k\to\infty}\int_{\|t\|\le N} \|w_k(\cdot,t) - w(\cdot,t)\|_{1,-\nu}dt = 0,$$

and

$$\|w_k(\cdot,t)\|_{1,-\nu} \le \frac{3}{2}\|w(\cdot,t)\|_{1,-\nu} \quad \text{a.e.}$$

Then by definition, $W = \lim_{k\to\infty}\int_{\|t\|\le N} w_k(\cdot,t)dt$ in $H^1_{-\nu}(\mathbb{R}^n)$.

Let $\psi \in C_0^\infty(\mathbb{R}^n)$. Then Fubini's theorem, the Lebesgue dominated convergence theorem, and the definitions of w_k and W imply

$$\int_{\mathbb{R}^n} u_0(x;N)e^{-\nu|x|}\overline{\psi(x)}dx = \int_{\|t\|\le N}\int_{\mathbb{R}^n} w(x,t)e^{-\nu|x|}\overline{\psi(x)}dxdt$$

$$= \lim_{k\to\infty}\int_{\|t\|\le N}\int_{\mathbb{R}^n} w_k(x,t)e^{-\nu|x|}\overline{\psi(x)}dx$$

$$= \int_{\mathbb{R}^n} W(x)e^{-\nu|x|}\overline{\psi(x)}dx. \qquad \square$$

References

[1] Archenback, J.D., A Theory of Elasticity with Microstructure for Directionally Reinforced Composites, Wien, New York, Springer-Verlag, 1975.

[2] Babuška, I., "Homogenization and its applications: Mathematical and computational problems," in Numerical Solution of Partial Differential Equations III, SYNSPADE 1975, B. Hubbard, ed., pp. 89-116, New York, Academic Press.

[3] Babuška, I., "Solution of interface problems by homogenization I, II, and III," SIAM J. Math. Anal., 7 (1976), pp. 603-634 and pp. 635-645, and 8 (1977), pp. 923-937.

[4] Babuška, I. and A.K. Aziz, "Survey lectures on the mathematical foundation of the finite element method," in The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, A.K. Aziz, ed., pp. 5-359, New York, Academic Press, 1973.

[5] Babuška, I. and I.C. Morgan, "Composites with a periodic structure: mathematical analysis and numerical treatment," Comp. & Math. with Applications, Oct. 1985.

[6] Babuška, I. and J.E. Osborn, Finite element methods for the solution of problems with rough input data, in Lecture Notes in mathematics, A. Dold, B. Echman, eds., No. 1121, 1983, pp. 1-18.

[7] Babuška, I. and J.E. Osborn, Generalized finite element methods, their performance and their rotation to mixed methods, SIAM J. Numer. Anal. 20 (1983), pp. 510-536.

[8] Bensoussan, A., J.L. Lions, and G. Papanicolaou, Asymptotic Analysis for Periodic Structures, Amsterdam, North-Holland, 1978.

[9] Burgers, J.M., "On a problem of homogenization," Quart. Appl. Math., 35 (1978), p. 421.

[10] Chams, C.C. and G.P. Sendeckyj, "Critique on theories predicting thermoelastic properties of fibrous composites," J. of Composite Materials, 2 (1968), pp. 332-358.

[11] Garg, S.K., V. Svalbonas, and G.A. Gurtman, Analysis of Structural Composite Materials, New York, Marcel Dekker, Inc., 1973.

[12] Hashin, Z., Theory of Fibers Reinforced Materials, NASA Report NASA CR-1974, Washington, D.C., 1972.

[13] Lions, J.L. and E. Magenes, Problèmes aux Limites Non Homogènes et Applications, vol. I, Paris, Dunod, 1968.

[14] Morgan, R.C., Mathematical aspects and computational considerations in the theory of homogenization, Ph.D. Dissertation, University of Maryland, 1982.

[15] Morgan, R.C. and I. Babuška, "An approach for constructing families of homogenized equations for periodic media: II: Properties of the kernel," Tech. Note BN-1095, Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742.

[16] Papanicolaou, G.C. and S.R.S. Varadhan, "Boundary value problems with rapidly oscillating random coefficients," Colloquia Matematica Socialatis, Janus Bolyai, Paper #27 in Random Fields, Esztergom Hungary, 1979.

[17] Rudin, W., Real and Complex Analysis, New York, McGraw-Hill, 1966.

[18] Sendeckyj, G.P., ed., Mechanics of Composite Materials, Vol. 2 of Composite Materials, eds., L.J. Broutman and R.H. Krock, New York, London, Academic Press, 1974.

[19] Yosida, K., Functional Analysis, 4th edition, Berlin, Springer-Verlag, 1974.

[20] Žhikov, V.V., S.M. Kozlov, O.A. Oleinik and Kha T'en Ngoan, "Averaging and G-convergence of differential operators," Russ. Math. Surveys, 34 (Spetember-October 1979), pp. 69-147.

The Laboratory for Numerical analysis is an integral part of the Institute for Physical Science and Technology of the University of Maryland, under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

- To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.

- To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.

- To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.

- To assist with the education of numerical analysts, especially at the postdoctoral level, in conjunction with the Interdisciplinary Applied Mathematics Program and the programs of the Mathematics and Computer Science Departments. This includes active collaboration with government agencies such as the National Bureau of Standards.

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Further information may be obtained from Professor I. Babuška, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742.

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